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SOME RESULT FIXED POINT THEOREMS IN C*-ALGEBRAS AND BANACH ALGEBRAS

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ABSTRACT: In this paper, the existence and uniqueness of coupled fixed point for mapping having the mixed monotone property in partially ordered metric spaces which endowed with vector-valued metrics are given.

***Keywords*:** POINT, ALGEBRAS, BANACH.

INTRODUCTION

Let X be a nonempty set. A mapping d: X × X $\rightarrow$ R^m is called a vector-valued metric on X if the following properties are satisfied:

(1) d(x, y) $\geq$ 0 for each x, y $\in$ X; if d(x, y) = 0, then x = y;
(2) d(x, y) = d(y, x) for each x, y $\in$ X;
(3) d(x, y) $\leq$ d(x, z) + d(z, y) for each x, y, z $\in$ X.

A set X equipped with a vector-valued metric d is called a generalized metric space and denoted by (X, d). By Mm,m(R^+) we mean that the set of all m×m matrices with positive elements. We denote by θ the zero matrix, and by I the identity m×m matrix. Let A $\in$ $M_{m,m}(R^+)$, A is said to be convergent to zero if and only if $A^n \rightarrow 0$ as n $\rightarrow\infty$ (for more details see [7]).

Let α, β $\in$ R^m, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, and c $\in$ R. By $\alpha \leq \beta$ (resp. $\alpha < \beta$) we mean that $\alpha_i \leq \beta_i$ (resp. $\alpha_i < \beta_i$) for each $1 \leq i \leq$ m, and by $\alpha \leq$ c (resp. α< c) for 1 $\leq$ i $\leq$ m.

Notice that for the proof of the main results, we need the following equivalent statements

(1) A is convergent towards zero;
(2) $A^n \rightarrow 0$ as n $\rightarrow\infty$;
(3) The eigenvalues of A are in the open uint disc, that is, $|\lambda| < 1$, for each $\lambda \in$ C with det(A−λ I) = 0;
(4) The matrix I − A is nonsingular and
$(I - A)^{-1} = I + A + \cdots + A^n + \cdots$;
(5) $A^n q \rightarrow 0$ and $qA^n \rightarrow 0$ as n $\rightarrow\infty$, for each q $\in$ R^m.

Where the proof of the above statements are the classical results in matrix analysis (for more details see [1], [5], and [6]).

Definition 1.1

(3). Let (X, $\leq$) be a partially ordered set and F : X × X $\rightarrow$ X. Mapping F is said to be has the mixed monotone property if F(x, y) is monotone non decreasing in x and is monotone non increasing in y, that is, for every x, y $\in$ X,

(i) for each $x_1, x_2 \in$ X, if $x_1 \leq x_2$, then $F(x_1, y) \leq F(x_2, y)$;
(ii) for each $y_1, y_2 \in$ X, if $y_1 \leq y_2$, then $F(x_1, y) \geq F(x_2, y)$.

Key words and phrases. Fixed points, Complete generalized metric space, fixed points.

Let (X, $\leq$) be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. The product space X × X is endowed with the following partial order:

for (x, y), (u, v) $\in$ X × X, (u, v) $\leq$ (x, y) $\Leftrightarrow$ x $\geq$ u, y $\leq$ v.

Definition 1.2 ([3]). Let (X, $\leq$) be a partially ordered set and F : X × X $\rightarrow$ X. An element (x, y) $\in$ X × X is said to be a coupled fixed point of the mapping F, if F(x, y) = x and F(y, x) = y.

Gnana Bhaskar and Lakshmikantham in [3], proved the following important Theorem:

Theorem 1.3. [3, Theorem 2.1]Let (X, $\leq$) be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let F : X × X $\rightarrow$ X be a continuous mapping having the mixed monotone property on X. Assume that there exists a k $\in$ [0, 1) with

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}\left[d(x,y),d(y,v)\right],$$

for all x $\geq$ u and y $\leq$ v. If there exist two elements x_0, y_0 $\in$ X with

$$x_0 \leq F(x_0,y_0) \quad \text{and} \quad y_0 \geq F(x_0,y_0)$$

Then there exist x, y $\in$ X such that

x = F(x, y) and y = F(y, x).

Definition 1.4. An element (x, y) $\in$ X × X is called

(1) a coupled coincidence point of mappings F : X × X $\rightarrow$ X and g : X $\rightarrow$ X if g(x) = F(x, y) and g(y) = F(y, x), and (gx, gy) is called a coupled point of coincidence.

(2) a common coupled fixed point of mappings F : X × X $\rightarrow$ X and g : X $\rightarrow$ X if x = g(x) = F(x, y) and y = g(y) = F(y, x).

Definition 1.5

(Let (X, $\leq$) be a partially ordered set and F : X × X $\rightarrow$ X and g : X $\rightarrow$ X be two self mappings. F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, if for all x_1, x_2 $\in$ X, $gx_1 \leq gx_2$ implies $F(x_1, y) \leq F(x_2, y)$ for any y $\in$ X, and for all y_1, $y_2 \in$ X, $gy_1 \geq gy_2$ implies $F(x, y_1) \leq F(x, y_2)$ for any x $\in$ X.

Definition 1.6

Let X be a non-empty set. We say that the mappings F : X ×X $\rightarrow$ X and g : X $\rightarrow$ X are commutative if g(F(x, y)) = F(gx, gy), for all x, y $\in$ X.

2. Main Results

Theorem 2.1

Let (X, $\leq$) be partial ordered Banach space, and F : X×X $\rightarrow$ X and g : X $\rightarrow$ X . and F mapping having the mixed g− monotone property on X. Assume that there exists A $\in$ $M_{m\times m}(R_+)$,A $\neq$ I be a nonzero matrix converging to zero whit:

(2.1) ||(F(x, y) − F(u, v))|| $\leq$ A[||gx − gu|| + ||gy − gv||],

for all x, y, u, v $\in$ X for which g(x) $\leq$ g(u) and g(v) $\leq$ g(y). Suppose that F(X × X) $\subset$ g(X), g is sequentially continuous and commutes with F and also suppose either F is continuous or X has the following property:

(I) if a non-decreasing {xn} $\rightarrow$ x, then $x_n \leq x$, for all n.

(II) if a non-decreasing {yn} $\rightarrow$ y, then y $\leq$ yn, for all n.

If there exist x_0, y_0 $\in$ X such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Then, there exists x, y $\in$ X such that g(x) = F(x, y) and g(y) = F(y, x), that is, F and g have coupled coincidence.

Proof. Let x_0, y_0 $\in$ X with $g(x_0) \leq F(x_0, y_0) = x_1$ and g(y0) $\leq$ $F(y_0, x_0) = g(x_1)$. Suppose that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Continuing this process, we have $g(x_n+1) = F(x_n, y_n)$ and $F(y_n, x_n) = g(x_n+1)$ for all n $\geq$ 0. Thus $g(x_n)$ $\leq$ $g(x_{n+1}$, and $g(y_{n+1})$. Therefore the g-monotone property of F implies

$g(x_{n+1}) = F(x_n, y_n) \leq F(x_n, y_n)$, and $F(y_n, x_n) = g(y_{n+1})$.

Thus $F(x_{n+1}, y_n) \leq F(x_{n+1}, y_{n+1}) = g(x_{n+2})$, $g(y_{n+2}) = F(y_{n+1}, x_{n+1}) \leq F(y_{n+1}, x_n)$.

Then we have $g(x_{n+1}) \leq g(x_{n+2})$ and $g(y_{n+2}) \leq g(y_{n+1})$. Therefore

$g(x_0) \leq g(x_1) \leq g(x_2) \leq \dots \leq g(x_n) \leq g(x_{n+1})$ _ . . . ,
and
$g(y_0) \leq g(y_1) \leq g(y_2) \leq \dots \leq g(y_n) \leq g(y_{n+1}) \geq \dots$.
We show that sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy:
$||g(x_n) - g(x_{n+1})|| = ||F(x_{n-1}, y_{n-1}) - F(x_n, y_n)||$

$$\leq \frac{A}{2}\left[\left\|g(x_{n-1} - g(x_n)\right\| + \left\|g(y_{n-1}) - g(y_n)\right\|\right]$$

$$\leq \frac{A^2}{2}\left[\left\|g(x_{n-2} - g(x_{n-1})\right\| + \left\|g(y_{n-2}) - g(y_{n-1})\right\|\right]$$

$\leq$ ...

$$\leq \frac{A^n}{2}\left[\left\|g(x_0 - g(x_n)\right\| + \left\|g(y_0) - g(y_{n-1})\right\|\right] \quad (2.2)$$

Similarly

$$\left\|g(y_n) - g(y_{n+1})\right\| = \left\|F(y_{n-1}, x_{n-1}) - F(y_n, x_n)\right\|$$

$$\leq A\left[\left\|g(y_{n-1} - g(y_n)\right\| + \left\|g(x_{n-1}) - g(x_n)\right\|\right]$$

$$\leq \frac{A}{2}\left[\left\|g(x_{n-1} - g(x_n)\right\| + \left\|g(y_{n-1}) - g(y_n)\right\|\right]$$

$$\leq \frac{A^2}{2}\left[\left\|g(y_{n-2} - g(y_{n-1})\right\| + \left\|g(x_{n-2}) - g(x_{n-1})\right\|\right]$$

$\leq$ ...

$$\leq \frac{A^n}{2}\left[\left\|g(y_0 - g(y_1)\right\| + \left\|g(x_0) - g(x_1)\right\|\right] \quad (2.3)$$

Together with (2.2) and (2.3) we have

$$\left\|g(x_n) - g(x_{n+1})\right\| + \left\|g(y_n) - g(y_{n+1})\right\| \leq A^n\left[\left\|g(x^0) - g(x_1)\right\| + \left\|g(y_0) - g(y_1)\right\|\right]$$

For $n > m$, we have
$||g(x_n) - g(x_m)|| + ||g(y_n) - g(y_m)||$

$\leq ||g(x_n) - g(x_{n-1})|| + ||g(y_n) - g(y_{n-1})|| + \dots + ||g(x_m) - g(x_{m+1})||$
$+||g(x_m) - g(x_{m+1})|| + ||g(y_m) - g(y_{m+1})||$

$\leq (A^{n-1} + A^{n-2} + \dots A^m)[||g(x_0) - g(x_1)|| + ||g(y_0) - g(y_1)||]$

(2.4) $\leq A^m (I - A)^{-1}[||g(x_0) - g(x_1)|| + ||g(y_0) - g(y_1)||$.

Thus sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy.
Since X is Banach algebra then these sequence are convergence. Thus there exists x, y $\in$ X such that

$$\lim_{n\to\infty} g(x_n) = x, and \lim_{n\to\infty} g(y_n) = y$$

By continuity of g, $\lim_{n\to\infty} g(g(x_{n+1}) = g(x)$ and $\lim_{n\to\infty} g(g(y_{n+1}) = g(y)$, and by commutativity of F and g, we have
$g(g(x_{n+1}) = g(F(x_n, y_n)) = F(g(x_n), g(y_n))$,
and
$g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g(y_n), g(x_n))$.
Now we show that $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Frits case

Let F be continuous.

$$g(x) = \lim_{n\to\infty} g(g(x_{n+1})) = \lim_{n\to\infty} F(g(x_n, g(y_n)) = F(\lim_{n\to\infty} g(x_n), \lim_{n\to\infty} g(y_n)) = F(x, y),$$

and

$$g(x) = \lim_{n\to\infty} g(g(x_{n+1})) = \lim_{n\to\infty} F(g(x_n, g(y_n)) = F(\lim_{n\to\infty} g(x_n), \lim_{n\to\infty} g(y_n)) = F(x, y),$$

Second case: Now, suppose that (I) and (II) hold. Since $g(x_n) \to x$ and $g(y_n) \to y$, then by (I) and (II), $g(x_n) \leq x$ and $y \leq g(yn)$ for all n. Thus

$$\|g(x) - F(x, y)\| \leq \|g(x) - g(g(x_{n+1}))\| + \|g(g(x_{n+1})) - F(x, y)\|$$
$$= \|g(x) - g(g(x_{n+1}))\| + \|F(g(x_n), g(y_n)) - F(x, y)\|$$
$$\leq \|g(x) - g(g(x_{n+1}))\| + \frac{A}{2}[\|g(g(x_N)) - g(x)\| + \|g(g(y_N) - g(y)\|]. \quad (2.5)$$

Hence, take the limit of both sides as $n \to 1$, we have $\|g(x)-F(x, y)\| \leq 0$. Thus g(x) = F(x, y) and similarly g(y) = F(y, x).

Theorem 2.2

Under the hypothesis of Theorem 2.1, suppose that for every (x, y), (x', y') $\in$ X × X, there exists a couple (u, v) $\in$ X × X such that (F(u, v), F(v, u)) and (F(x', y'), F(y', x')). Then F and g have a unique couple common fixed point, in other word, there exists a unique (x, y) $\in$ X × X such that x = g(x) = F(x, y), and y = g(y) = F(y, x).

Proof. Existence of the set of coupled coincidence points is due to theorem 2.1. Let (x, y), (x', y') $\in$ X×X, be the coupled coincidence points, that is g(x) = F(x, y), g(y) = F(y, x) and g(x') = F(x', y'), F(y', x') = g(y'). By assumption, there is a couple (u, v) $\in$ X × X such that (F(u, v), F(v, u)) is comparable to (F(x, y), F(y, x)) and (F(x', y'), F(y', x')). Set $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ with $g(u_1) = F(u_0, v_0)$, $g(v_1) = F(v_0, u_0)$.

Similar to the proof of theorem 2.1, we construct the sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ in the way that $g(x_{n+1} = F(u_n, v_n)$, $g(v_{n+1} = F(v_n, u_n)$. Similarly we can construct the the sequences $\{g(x_n)\}$, $\{g(y_n)\}$, $\{g(x'_n)\}$ and $\{g(y'_n)\}$:

$X_0 = x \Rightarrow g(x_{n+1}) = F(x_n, y_n)$,

$y_0 = y \Rightarrow g(y_{n+1}) = F(y_n, x_n)$,

and

$$y'_0 = y' \Rightarrow g(y'_{n+1}) = F(y'_n, x'_n)$$

Since (g(x), g(y)) = (F(x, y), F(y, x)) = $(g(x_1), g(y_1))$ and (F(u, v), F(v, u)) = $(g(u_1), g(v_1))$ are comparable, then $g(x) \leq g(u_1)$ and $g(v_1) \leq g(y)$. Similarly (g(x), g(y)) and $(g(u_1), g(v_1))$ are comparable, that is $g(x) \leq g(u_n)$ and $g(v_n) \leq g(y)$, for $n \geq 1$,

$$\|g(x) - g(u)\| = \|F(x, y) - F(u_n, v_n)\| \leq \frac{A}{2}[\|g(x) - g(u_n)\| + \|g(y) - g(v_n)\|],$$

and

$$\|g(y) - g(u_{n+1})\| = \|F(y,x) - F(u_n, u_n\| \leq \frac{A}{2}[\|g(y) - g(u_n)\| + \|g(x) - g(u_n)\|]$$

Which imply that

$$\|g(x) - g(u_{n+1})\| + \|g(y) - g(v_{n+1})\| \leq A[\|g(x) - g(u_n)\| + \|g(y) - g(v_n)\|].$$

Thus

$$\|g(x) - g(u_{n+1})\| + \|g(y) - g(u_{n+1})\| \leq A^n[\|g(x) - g(u_1)\| + \|g(y) - g(v_1)\|].$$

If $n \to \infty$ then $A^n \to 0$, then $\|g(x) - g(u_{n+1})\| + \|g(y) - g(v_{n+1})\| \to 0$. Therefore

$$\lim_{n\to\infty}\|g(x) - g(u_{n+1})\| = 0, and \lim_{n\to\infty}\|g(y) - g(u_{n+1})\| = 0$$

Similarly

$$\lim_{n\to\infty}\|g(x') - g(u_{n+1})\| = 0, and \lim_{n\to\infty}\|g(y') - g(u_{n+1})\| = 0$$

Thus

$\|g(x) - g(x')\| \leq \|g(x) - g(u_{n+1})\| + \|g(u_{n+1}) - g(x')\| \to 0$ as $n \to \infty$,

$\|g(y) - g(y')\| \leq \|g(y) - g(v_{n+1})\| + \|g(v_{n+1}) - g(y')\| \to 0$ as $n \to \infty$.

Therefore g(x) = g(x') and g(y) = g(y'). By of commutativity of F and g with g(x) = F(x, y) and g(y) = F(y, x), we get

g(g(x)) = g(F(x, y)) = F(g(x), g(y)),
and
g(g(y)) = g(F(y, x)) = F(g(y), g(x)).

By letting t = g(x) and s = g(y), then g(t) = F(t, s) and g(s) = F(s, t). This means that (t, s) is coupled coincidence point, also g(x) = g(t) and g(y) = g(s), where t = x' and s = y'.
Since t = g(x) and s = g(y), then g(t) = t and g(s) = s. So (t, s) is coupled common fixed point of F and g.

Uniqueness, follows form g(x) = g(x') and g(y) = g(y'). Indeed, for another coupled common fixed point (¯t, ¯s) of F and g, then ¯t = g(¯t) = g(t) = t and ¯s = g(¯s) = g(s) = s.
We now present some results in C^*-algebras.

Theorem 2.3

Let A be a unital C^*-algebra and let F : $\Omega_{A\times A} \subseteq A\times A \to \Omega_A$ be a holomorphic map that satisfies the conditions F(0, 0) = 0, $\frac{\partial F}{\partial X}$ (0, 0) = idA, $\frac{\partial F}{\partial y}$ (0, 0) = 0, $\frac{\partial^2 f}{\partial x^2}$ (0,0)=0, $\frac{\partial^2 f}{\partial y^2}$ (0,0)=0, and $\frac{\partial^2 f}{\partial y \partial x}$ (0,0)=0. Then every (a, b) $\in \Omega_{A\times A} \cap Z$ (A×A) is a coupled fixed point for F. Furthermore (a^*, b^*) is a coupled fixed point of F.
Proof. Since every unital C^*-algebra is semisimple ([8, Corollary 3.2.13]), so by Theorem 3.1 of [9], every (a, b) $\in \Omega_{A\times A} \cap$ Z(A × A) is a coupled fixed point for F. Now, suppose x = (a, b), y = (a', b') $\in$ A × A.
Since $||x|| = ||x^*||$, therefore if x $\in \Omega_{A\times A} \cap Z_{A\times A}$ then $x^* \in \Omega_{A\times A} \cap Z_{A\times A}$. As well as, $(x^* y)^* = y^* x = xy^* = (yx^*)^*$ that is $x^*y = yx^*$.

Theorem 2.4

Let A be a unital C^*-algebra, let F : $\Omega_{A\times A} \subseteq A\times A \to \Omega_A$, and let g : $\Omega_A \to \Omega_A$ such that F has the mixed g-monotone property. Assume g is biholomorphic function from Ω_A into Ω_A such that g(0) = 0 and g'(0) = id_A. Then g is *-preserving on $\Omega_A \cap Z_A$
Proof. Let x $\in \Omega_A \cap Z_A$,by theorem (2.3) $x^* \in \Omega_A \cap Z_A$ and theorem of [10], g($\Omega_A \cap Z_A$)=$\Omega_A \cap Z_A$ we have $g(x^*) = x^* = g(x)^*$.

REFERENCES

Du WS. 2010. Coupled fixed point theorems for nonlinear contractions satisfied MizoguchiTakahashis condition in quasiordered metric spaces, Fixed Point Theory Appl. 2010, Article ID 876372, 9 pages.

Dales HG. 2000. Banach algebras and automatic continuity, London Math. Society Monographs, Volume 24, Clarendon Press, Oxford.

Filip AD and sel P. 2010. Fixed point theorems on spaces endowed with vector-valued metrics, Fixed Point Theory and Applications, vol. 2010, Article ID 281381, 15 pages.

Gnana Bhaskar T, Lakshmikantham V. 2006. Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65, 1379-1393.

laire G and Kaber SM. 2008. Numerical Linear Algebra, vol. 55 of Texts in Applied Mathematics, Springer-New York.

Precup R. 2009. The role of matrices that are convergent to zero in the study of semi linear operator systems, Mathematical and Computer Modelling, vol. 49, no. 3-4, pp. 703-708.

Rus IA. 1979. Principles and Applications of the Fixed Point Theory, Dacia, Cluj-Napoca, Romania.

Varga RS. 2000. Matrix Iterative Analysis, vol. 27 of Springer Series in Computational Mathematics, Springer-Berlin.

Zohri A, Jabbari A. 2012. Generalization of some properties of Banach algebras to fundamental locally multiplicative topological algebras, Turk J. Math. 36(2012), 445-451.